Introduction to Dynamical Systems

Solutions Problem Set 5

Exercise 1. Let (X, m) be a measure space with $m(X) = 1$, and let $T \colon X \longrightarrow X$ be measure-preserving. Recalling the notation from last lecture, set $U_T f = f \circ T$. Then show that T is ergodic if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \left\langle U_T^j f, g \right\rangle = \langle f, 1\rangle \cdot \langle 1, g\rangle$$

for all $f, g \in L^2(X, m)$.

Solution. Assuming that T is ergodic and using the mean ergodic theorem of Von Neumann, we find that the ergodic averages of f converge to a constant, which immediately yields the result.

Assume now that the convergence in the statement holds. Then, given $A, B \subset X$ with $m(A)m(B) > 0$, set $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$, and apply the mean ergodic theorem to find that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} m(T^{-j}(A) \cap B) = m(A)m(B).$$

This implies that there is $n \geq 1$ such that $m(T^{-n}(A) \cap B) > 0$, which in particular characterizes the fact that T is ergodic. □

Exercise 2. Let (X, m) a finite measure space, $T \colon X \longrightarrow X$ measure-preserving, and let $f \in L^p(X, m)$, $1 \leq p < \infty$. Then show that there is $f^* \in L^p(X, m)$, invariant under composition with T, and such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) = f^*(x)$$

with the convergence in the L^p-topology.

Solution. Notice that for $1 \leq p < q \leq \infty$, the inclusion $L^q \subset L^p$ holds true, as

$$\|f\|_{L^p} \leq m(X)^{1/r} \|f\|_{L^q}, \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{q}.$$

We argue similarly as in **Exercise 4** in Problem Set 4. That is, for $1 \leq p < 2$, we can replicate that proof step by step using the density of L^2 in L^p and the inequality above when $q = 2$. On the other hand, if $p > 2$, we need to work ourselves around the estimate above by exploiting the density of L^∞ in L^p. In fact, given $f \in L^\infty$, we find

$$\|f\|_{L^p} \leq \|f\|_{L^\infty}^{(p-2)/p} \|f\|_{L^2}^{2/p}.$$

Now, given $\varepsilon > 0$ and choosing N, M large enough so that

$$\left\| \frac{1}{N} \sum_{j=0}^{N-1} f \circ T^{-j} - \frac{1}{M} \sum_{j=0}^{M-1} f \circ T^{-j} \right\|_{L^2}^{2/p} < \frac{\varepsilon}{1 + (2\|f\|_{L^\infty})^{(p-2)/p}},$$

we may achieve the result for $f \in L^\infty$ through the same argument as in the previous problem set. We conclude for $f \in L^p$ by density. □

Exercise 3. Let X be a finite set and let $\sigma\colon X \longrightarrow X$ a permutation. We say σ is cyclic provided that X is an orbit of σ. Show that if $f\colon X \longrightarrow \mathbb{R}$ and σ is a cyclic permutation, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(\sigma^j x) = \frac{1}{|X|} \sum_{x\in X} f(x).$$

Solution. The exercise follows from an application of the mean ergodic theorem, after making sure that σ is ergodic with respect to the counting measure m on X. Notice that it is measure-preserving since it is a bijection, and X being an orbit means that every orbit is periodic and equal to X itself. In particular, for any $A \subset X$ with $m(A) > 0$ there is $x \in A$, and therefore

$$m\left(\bigcup_{j\geq 1} \sigma^{-n}(A)\right) = m(X) = |X|,$$

and σ is ergodic. □